ALGEBRAIC TOPOLOGY I WS23/24, HOMEWORK SHEET 7

DEADLINE: FRIDAY, DECEMBER 1ST

Problem 1. Let $n \geq 2$, and choose generators $\widetilde{x} \in H^n(S^n, \mathbb{Z})$ and $\widetilde{y} \in H^{2n}(S^{2n}, \mathbb{Z})$. Now let $f : S^{2n-1} \to S^n$ be a continuous map with mapping cone $C(f)$. We obtain generators $x \in H^n(C(f), \mathbb{Z})$ and $y \in H^{2n}(C(f), \mathbb{Z})$ from the isomorphisms $H^n(C(f), \mathbb{Z}) \xrightarrow{\cong} H^n(S^n, \mathbb{Z})$ and $H^{2n}(S^{2n}, \mathbb{Z}) \xrightarrow{\cong} H^{2n}(C(f), \mathbb{Z})$ induced by the inclusion of the n-skeleton $S^n \to C(f)$ and the projection to the $2n$-cell $C(f) \to S^{2n}$, respectively. We then define the Hopf invariant $h(f)$ to be the unique integer such that $x^2 = h(f)y$.

(1) Show that $h(f) = 0$ if n is odd.
(2) Show that the Hopf invariant gives rise to a group homomorphism $h : \pi_{2n-1}S^n \to \mathbb{Z}$.
(3) Show that if $g \colon S^n \to S^n$ has degree d, then $h(g \circ f) = d^2 h(f)$.
(4) Consider the composite

$$\alpha : S^{2n-1} \to S^n \vee S^n \to S^n,$$

where the first map is the attaching map of the $2n$-cell of $S^n \times S^n$. Show that $h(\alpha)$ is ± 2.
(5) The space ΩS^{n+1} is homotopy-equivalent to a CW complex with one cell in every dimension a multiple of n (you don't have to prove this). What is the Hopf invariant of the attaching map of the $2n$-cell, up to a sign?

Problem 2. Let $\pi_*^s X$ denote the stable homotopy groups of a space X. Construct natural long exact sequences of the form

$$\ldots \to \pi_n^s A \to \pi_n^s X \to \pi_n^s X/A \to \pi_{n-1}^s X \to \ldots$$

for pointed CW-pairs (X, A).

(Hint: Let $i \colon A \to X$ denote the inclusion. Construct a natural map $S^1 \wedge hofib_x(i) \to C(i)$ and show that it is highly-connected if A and X are.)